Acta Sci. Math., 51 (1987), 295—308

Pseudomodular lattices and continuous matroids

A. BJÖRNER*) and L. LOVÁSZ

Dedicated to the memory of András Huhn

0. Introduction

If $(E, \mathcal{M})$ is a linear matroid (i.e., represented by a subset of vectors in a linear space) then $(E, \mathcal{M})$ can be embedded in the full linear matroid (the matroid formed by all vectors in that linear space) in a natural way. The most significant property of the full linear matroids is that the lattice of their flats is modular. (In fact, apart from direct sums, loops and parallel elements and the non-desarguesian projective planes, this property characterizes full linear matroids.)

Other classes of matroids like graphic, algebraic and transversal matroids also have natural "full" members, which are, however, non-modular. For the case of full algebraic matroids, INGLETON and MAIN [7] proved that the following property (strictly weaker than modularity) still holds: any three lines such that any two are coplanar, but all three are not coplanar, have a point in common. LINDSTRÖM [11]—[13] observed that this fact is a basic property of full algebraic matroids, and used it to prove that several other geometric results on projective spaces, e.g. Desargues's Theorem, also carry over to full algebraic matroids.

DRESS and LOVÁSZ [4] proved various generalizations of the Ingleton—Main Lemma, and showed that one of them suffices to extend the minimax formula for matchings in linear matroids (LOVÁSZ [14]) to algebraic matroids. It was observed that full graphic matroids (or, equivalently, partition lattices) and full transversal matroids also have this property. Another related property, the existence of "pseudo-intersections", was established for the full algebraic matroids.

Modularity of the subspace lattice of linear spaces also plays a crucial role in a contruction, due to VON NEUMANN [16], of "continuous geometries". To obtain

Received September 1, 1986, and in revised form March 25, 1987.

*) Partially supported by the National Science Foundation.

these, one embeds the subspace lattice of the n-dimensional linear space over some field F in the subspace lattice of the nq-dimensional linear space over the same field, so that any flat x of rank $r(x)$ is mapped onto a flat of rank $q \cdot r(x)$. Such a "stretch embedding" which preserves meet and join can be constructed using the modularity of the lattices.

A construction of a "continuous partition lattice" based on stretch embeddings was given by Björner [2]. This construction depends on the fact that partition lattices have sufficiently many modular elements. A general scheme to obtain continuous analogues of sequences of geometric lattices was also outlined: the scheme depends on the existence of "stretch embeddings" between these lattices.

The main result of this paper is that the existence of "pseudointersections" in suitable sequences of geometric lattices can be used to construct stretch embeddings and thereby continuous analogues. In particular, we construct continuous transversal geometries, continuous algebraic geometries over any field, and obtain a new theoretical explanation for the existence of the continuous partition lattice.

Semimodular lattices with pseudointersections, which we call *pseudomodular*, seem to be worth studying even without an eye on continuous geometries. We shall show that such lattices arise in the study of antimatroids (abstract convexity spaces). In fact, an antimatroid with Caratheodory number 2 has a pseudomodular lattice of feasible sets.

1. Pseudomodular lattices

In this paper we shall assume some modest familiarity with lattices and matroids. For details concerning these notions see Birkhoff [1] and Welsh [17], respectively.

Let L be a semimodular lattice. We assume without further mention that all semimodular lattices considered have finite rank. Let $r(x)$ denote the rank function of L. For each $x, y \in L$, we set

$$P_{x,y} = \{z \leqq y\colon\ r(x \vee z) - r(z) = r(x \vee y) - r(y)\}.$$

Note that it would suffice to require that $r(x \vee z) - r(z) \leqq r(x \vee y) - r(y)$ in this definition, since the reverse inequality is always true by the submodularity of the rank function.

This set lies in the interval $[x \wedge y, y]$. To see this, let $z \in P_{x,y}$. Then $z \leqq (x \vee z) \wedge y$ and hence by the submodularity of the rank function,

$$\begin{aligned} r(z) \leqq r\big((x \vee z) \wedge y\big) \leqq r(x \vee z) + r(y) - r\big((x \vee z) \vee y\big) = \\ = r(x \vee z) + r(y) - r(x \vee y) = r(z). \end{aligned}$$

So $z = (x \vee z) \wedge y \geqq x \wedge y$.

Clearly, $P_{x,y}$ is a filter in the interval $[x\wedge y, y]$, i.e., if $z\in P_{x,y}$ and $z\leqq u\leqq y$ then $u\in P_{x,y}$.

If the set $P_{x,y}$ has a unique least element then we call this the *pseudointersection* of x and y and denote it by $x\urcorner y$. The lattice L is called *pseudomodular* if every pair of its elements have a pseudointersection.

Note that in general $x\urcorner y\neq y\urcorner x$ (cf. Lemma 1.1 below). Furthermore, the existence of a pseudointersection is not a symmetric relation. For an example of this, take three pairwise parallel lines in affine 3-space, not all in a plane. Let y denote one of these lines and x, the plane spanned by the other two. Then in the geometric lattice formed by the points of these lines, $x\urcorner y$ exists but $y\urcorner x$ does not.

The relationship between the pseudointersection and the (ordinary) intersection of two lattice elements is illuminated by the following lemma.

Lemma 1.1. *For any two elements x and y in a semimodular lattice L, the following are equivalent:*

(i) *x and y form a modular pair, i.e.,* $r(x\vee y)+r(x\wedge y)=r(x)+r(y)$.

(ii) *$x\urcorner y$ exists and $x\urcorner y\leqq x$.*

(iii) *$x\urcorner y$ exists and $x\urcorner y=x\wedge y$.*

(iv) $x\wedge y\in P_{x,y}$.

Proof. All implications (i)→(iv)→(iii)→(ii)→(i) are straightforward.

The following lemma gives some means to verify the existence of pseudointersections.

Lemma 1.2. *For any two elements x and y of a semimodular lattice L, the following are equivalent:*

(i) *$x\urcorner y$ exists, i.e., $P_{x,y}$ has a unique least element.*

(ii) *$P_{x,y}$ is closed under meets.*

(iii) *If $u, v, z\in P_{x,y}$ and z covers u and v, then $u\wedge v\in P_{x,y}$.*

Proof. The only non-trivial implication is that (iii)→(i). To verify this, assume, by way of contradiction, that a and b are distinct minimal elements of $P_{x,y}$, and choose a and b so that $a\vee b$ is as low in the lattice as possible. Let u be an element in the interval $[a, a\vee b]$ covered by $a\vee b$ and let v be an element in the interval $[b, a\vee b]$ covered by $a\vee b$. Then by (iii), $u\wedge v\in P_{x,y}$. Let c be a minimal element of $P_{x,y}$ below $u\wedge v$, then $a\vee c\leqq u<a\vee b$ and $b\vee c\leqq v<a\vee b$. Since c is distinct from at least one of a and b, this contradicts the choice of a and b. (This lemma in fact holds for any filter in any interval of any lattice of finite length.)

It will be useful to remark that the assertion (iii) in Lemma 1.2 holds automatically if u, v is a modular pair, i.e., if u covers $u\wedge v$. For, by submodularity and the definition of $P_{x,y}$, we have the following:

$$1 = r(u)-r(u\wedge v) \geqq r(u\vee x)-r((u\wedge v)\vee x) \geqq$$
$$\geqq r(z\vee x)-r(v\vee x) = r(z)-r(v) = 1.$$

So equality must hold throughout, and equality in the first inequality means just that $u\wedge v\in P_{x,y}$.

Lemma 1.3. *Let L be a geometric lattice and $x, y\in L$. If $x\urcorner y$ exists then it is equal to the meet of all $z\in L$ such that y covers z and $x\vee y$ covers $x\vee z$.*

Proof. Let $t=\wedge\{z\colon y$ covers z and $y\vee x$ covers $z\vee x\}$. Note that the second condition on z is equivalent to $z\in P_{x,y}$. Hence $t\in P_{x,y}$ and thus $t\geqq x\urcorner y$. On the other hand, the interval $[x\urcorner y, y]=P_{x,y}$ is a geometric lattice and hence its bottom element is the meet of its coatoms. This proves that $t=x\urcorner y$.

The existence of pseudointersections can be characterized by the non-existence of certain configurations in the lattice. Such a result is stated in the following theorem.

Theorem 1.4. *Let L be any semimodular lattice. Then the following are equivalent:*

(i) *L is pseudomodular.*

(ii) *Let $a, b, c\in L$, and assume that $r(a\vee c)-r(a)=r(b\vee c)-r(b)=r(a\vee b\vee c)-r(a\vee b)$. Then $r((a\vee c)\wedge(b\vee c))-r(a\wedge b)=r(a\vee c)-r(a)$.*

(iii) *Let $x, y, z\in L$ and assume that x covers $x\wedge z$ and y covers $y\wedge z$. Then $r(x\wedge y)-r(x\wedge y\wedge z)\leqq 1$.*

(iv) *Let $x, y, z, u\in L$, and assume that u covers x, y and z, and z covers $x\wedge z$ and $y\wedge z$. Then $r(x\wedge y)-r(x\wedge y\wedge z)\leqq 1$.*

(v) *Let $x, y, z, u\in L$, and assume that u covers x and y, $z\leqq u$, and z covers $x\wedge z$ and $y\wedge z$. Then $r(x\wedge y)-r(x\wedge y\wedge z)\leqq r(u)-r(z)$.*

Remark. Property (ii) has the following consequences. Since $c\leqq(a\vee c)\wedge(b\vee c)$, it implies that $r(c)-r(a\wedge b)\leqq r(a\vee c)-r(a)$. Also, it follows that a and $(a\vee c)\wedge(b\vee c)$ form a modular pair and $a\wedge(a\vee c)\wedge(b\vee c)=a\wedge b$. Hence, $a\wedge c=a\wedge(a\vee c)\wedge(b\vee c)\wedge c=a\wedge b\wedge c$. Similarly, $b\wedge c=a\wedge b\wedge c$. It also follows that $a\wedge c\leqq a\wedge b$.

Proof. (i)→(ii): Let $d=(a\vee c)\wedge(b\vee c)$. Then $a\in P_{d,a\vee b}$ since $a\vee c=a\vee d$ and $a\vee b\vee c=a\vee b\vee d$, and so $r(a\vee d)-r(a)=r(a\vee b\vee d)-r(a\vee b)$ by the hypothesis in (ii). Similarly $b\in P_{d,a\vee b}$ and hence by the pseudomodularity of L, $a\wedge b\in P_{d,a\vee b}$. This means that $r((a\wedge b)\vee d)-r(a\wedge b)=r(a\vee b\vee d)-r(a\vee b)$. Since $(a\wedge b)\vee d=d$ and $a\vee b\vee d=a\vee b\vee c$, this implies the assertion of (ii).

(ii)→(iii): We may assume that $z=(x\wedge z)\vee(y\wedge z)$; if this is not already the case we can just let $(x\wedge z)\vee(y\wedge z)$ play the role of z without changing the situation.

We may also assume that $x\wedge y\not\leqq z$ (since otherwise $x\wedge y=x\wedge y\wedge z$), and that $x\neq y$. It follows that $x\wedge z\neq y\wedge z$, and $x=(x\wedge z)\vee(x\wedge y)$, $y=(y\wedge z)\vee(x\wedge y)$. Also, $r(z)<r(z\vee(x\wedge y))\leqq r(z\vee x)\leqq r(z)+r(x)-r(z\wedge x)=r(z)+1$. Hence, $z\vee(x\wedge y)$ covers z. Now letting $a=x\wedge z$, $b=y\wedge z$ and $c=x\wedge y$ in (ii), assertion (iii) follows.

(iii)→(iv) An easy special case.

(iv)→(v): We prove this by induction on $r(u)-r(z)$; (iv) is just the special case of (v) when this difference is 1. We may assume that $x\wedge y\wedge z\neq x\wedge y$. Let p be an element of the interval $[x\wedge y\wedge z, x\wedge y]$ covering $x\wedge y\wedge z$. Then clearly $p\not\leqq z$ and so $p\not\leqq x\wedge z$ and $p\not\leqq y\wedge z$. Hence $v=z\vee p$ covers z by submodularity and similarly, $(z\wedge x)\vee p$ covers $z\wedge x$ and $(z\wedge y)\vee p$ covers $z\wedge y$. Clearly $(z\wedge x)\vee p\leqq\leqq v\wedge x<v$ and hence $v\wedge x=(z\wedge x)\vee p$. Hence $v\wedge x$ is covered by v and similarly, $v\wedge y$ is also covered by v. Applying (iv) with v, $v\wedge x$, $v\wedge y$ and z in place of u, x, y and z we obtain that

$$r(x\wedge y\wedge v)-r(x\wedge y\wedge z)\leqq 1.$$

Applying the induction hypothesis with u, x, y and v in place of u, x, y and z we obtain that

$$r(x\wedge y)-r(x\wedge y\wedge v)\leqq r(u)-r(v) = r(u)-r(z)-1.$$

This proves (v).

(v)→(i): We verify Lemma 1.2(iii). Let $u, v, z\in P_{x,y}$, where z covers both u and v. Then by the definition of $P_{x,y}$, $z\vee x$ covers $u\vee x$ and $v\vee x$, and $z\wedge(u\vee x)=u$, $z\wedge(v\vee x)=v$. So (v) can be applied with $u\vee x$, $v\vee x$, z and $z\vee x$ in place of x, y, z and u, and we obtain that

$$r((u\vee x)\wedge(v\vee x))-r((u\vee x)\wedge(v\vee x)\wedge z)\leqq r(z\vee x)-r(z).$$

Since, as remarked, $(u\vee x)\wedge(v\vee x)\wedge z=u\wedge v$, this implies that

$$r((u\wedge v)\vee x)-r(u\wedge v)\leqq r((u\vee x)\wedge(v\vee x))-r(u\wedge v)\leqq r(z\vee x)-r(z),$$

which proves that $u\wedge v\in P_{x,y}$.

Lindström [13] proved the following generalization of the Ingleton—Main Lemma for full algebraic matroids: if a, b and c are three flats such that $r(a)=r(b)=r(c)=n$, $r(a\vee b)=r(a\vee c)=r(b\vee c)=n+1$ and $r(a\vee b\vee c)=n+2$ then $a\wedge b=a\wedge c=b\wedge c=a\wedge b\wedge c$ and $r(a\wedge b\wedge c)=n-1$. This follows immediately from property (iv) in the above theorem. He conjectured that if a, b and c are three flats in an algebraic matroid such that $r(a)=r(b)=r(c)=n$, $r(a\vee b)=r(a\vee c)=r(b\vee c)=n+k$ and $r(a\vee b\vee c)=n+2k$ then $a\wedge b=a\wedge c=b\wedge c=a\wedge b\wedge c$ and $r(a\wedge b\wedge c)=n-k$. This conjecture follows from the inequality in (ii) easily.

2. Examples of pseudomodular lattices

In this section we discuss some classes of semimodular lattices which have pseudointersections. We start with two obvious examples:

Example 1. Modular lattices.

Example 2. Semimodular lattices of length at most 3.

Next we discuss three families of geometric lattices, (i.e., matroids) which have pseudointersections. These are "full" members in their own class (algebraic matroids, graphic matroids, transversal matroids) in a very natural way. The "full" linear matroids, i.e., linear spaces, have a modular subspace lattice and hence they are covered by Example 1. It would be important to understand the structure of those classes of matroids which have natural "full" members and why these full members tend to be pseudomodular.

Example 3. Full algebraic matroid lattices. These can be described as follows: let F and K be algebraically closed fields and $F\subset K$. Then the algebraically closed subfields of K containing F form a geometric lattice, which we denote by $\mathscr{L}(F,K)$.

The fact that $\mathscr{L}(F,K)$ has pseudointersections was shown by Dress and Lovász [4]. For the sake of completeness, we describe the simple construction of the operation $\urcorner$. So let X and Y be two algebraically closed fields with $F\subset X$, $Y\subset K$. Let $\{x_1, \ldots, x_m\}$ be a transcendence basis of X over F. Consider the ideal I of all polynomials over Y in m variables which are satisfied by $(x_1, \ldots, x_m)$, and a basis $q_1, \ldots, q_N$ of this ideal. We may assume that each q_i has at least one coefficient that is equal to 1. Then the algebraically closed subfield T of Y generated by the coefficients of $q_1, \ldots, q_N$ is the pseudointersection of X and Y.

Example 4. Partition lattices, i.e., circuit matroids of complete graphs. We show that the lattice of partitions of a set E has pseudointersections, using Lemma 1.2(iii). Assume that u, v and z are three partitions in $P_{x,y}$, and that z covers both u and v, i.e., both u and v arise from z by splitting a partition class into two. The fact that $u,v,z\in P_{x,y}$ implies that $r(x\vee z)-r(z)=r(x\vee u)-r(u)=r(x\vee v)-{}$ $-r(v)=r(x\vee y)-r(y)$, and hence $r(x\vee u)=r(x\vee v)=r(x\vee z)-1$. We want to show that $r(x\vee u)-r\big(x\vee(u\vee v)\big)=r(u)-r(u\wedge v)$.

By the remark after Lemma 1.2, the only non-trivial case to consider is when u and v do not form a modular pair, i.e., when they arise from z by splitting the same class A in two different ways $A'_u\cup A''_u$ and $A'_v\cup A''_v$ so that the intersections $B_1=A'_u\cap A'_v$, $B_2=A'_u\cap A''_v$, $B_3=A''_u\cap A'_v$ and $B_4=A''_u\cap A''_v$ are all non-empty. So $r(u)-r(u\wedge v)=2$, and submodularity implies that $r(x\vee u)-r\big(x\vee(u\wedge v)\big)\leqq 2$. Now if $r(x\vee u)-r\big(x\vee(u\wedge v)\big)\leqq 1$ then the sets B_1, B_2, B_3 and B_4 cannot belong to different classes in $x\vee(u\wedge v)$ and hence there exists a sequence $x_1, \ldots, x_k$ of elements of E such that $x_1 \in B_i$, $x_n \in B_j$ $(i \neq j)$, no other member of the sequence belongs to A, and any two consecutive members of the sequence are either in one class of x or in one class of $u \wedge v$. Without loss of generality we may assume that $B_i \subset A'_u$ and $B_j \subset A''_u$. But then B_i and B_j must belong to the same class of $x \vee u$, which is a contradiction.

Example 5. Full transversal matroids. The full transversal matroid $\mathcal{TM}(r)$ of rank r is defined as follows. First we construct a bipartite graph G. Let S be a set with r elements; this will be one of the color classes. For each subset $S' \subseteq S$, we take denumerably infinitely many new vertices and connect them by edges to the vertices in S'. The set T of these new vertices will be other color class. Now $\mathcal{TM}(r)$ is defined as the transversal matroid induced by G on T. So a set $T' \subseteq T$ is independent iff G contains a matching covering T'.

Using König's Theorem, it is easy to show that the flats in $\mathcal{TM}(r)$ have the following structure: take a set $A \subseteq S$, and also a set $B \subseteq T$ such that every non-empty subset $B' \subseteq B$ has at least $|B'|+1$ neighbors in $S-A$. Let $\Omega(A)$ denote the set of points x in T such that all neighbors of x are in A. Then $F(A, B)=\Omega(A) \cup B$ is a flat of rank $|A|+|B|$ in $\mathcal{TM}(r)$, and every flat is of this form.

The pseudomodularity of full transversal matroids (in fact, of a much larger class of transversal matroids) will follow from the results in the next section.

Example 6. Antimatroids with Caratheodory number 2. Antimatroids were introduced by EDELMAN [5] and JAMISON-WALDNER [8] as combinatorial abstractions of convex sets. For our purposes, the following definition will suffice. Let E be a finite set and $\mathcal{F}$, a family of subsets of E with the following properties:

a) if $X \in \mathcal{F}$ and $Y \in \mathcal{F}$ then $X \cup Y \in \mathcal{F}$;

b) if $X \in \mathcal{F}$, $X \neq \emptyset$ then there exists an element $x \in X$ such that $X-x \in \mathcal{F}$.

Then the pair $(E, \mathcal{F})$ is called an *antimatroid*. The members of $\mathcal{F}$ are called *feasible sets*, their complements are called *convex sets*. Since the family of convex sets is closed under intersection, we can define the *convex hull* of any subset X of E as the intersection of all convex supersets of X. These notions share many of the combinatorial properties of convex sets in the usual sense. We shall need the following two elementary facts: (I) if X and Y are feasible and $Y \not\subset X$, then there exists an element $y \in Y-X$ such that $X \cup y$ is feasible; (II) p is in the convex hull of G if and only if every feasible set containing p has a non-empty intersection with G.

We define the *Caratheodory number* of an antimatroid as the least integer k with the following property: whenever an element p is contained in the convex hull of a set G, it is also contained in the convex hull of some subset $G' \subset G$ with $|G'| \leqq k$. In the language of KORTE and LOVÁSZ [9], the Caratheodory number is one less than the maximum size of a circuit of the antimatroid. For various properties of antimatroids, see also EDELMAN and JAMISON [6], KORTE and LOVÁSZ [10], BJÖRNER, KORTE and LOVÁSZ [3].

The feasible sets of an antimatroid of $\mathscr{F}$ form a semimodular lattice $\mathscr{L}(E, \mathscr{F})$ under ordinary inclusion. More strongly, $\mathscr{L}(E, \mathscr{F})$ is *locally free* (i.e., the elements covering any given element generate a Boolean subalgebra) and every locally free semimodular lattice has a unique representation as the feasible set lattice of an antimatroid (EDELMAN [5]). The rank of any element $X \in \mathscr{F}$ in this lattice is just its cardinality.

It was proved by KORTE and LOVÁSZ [9] that if the Caratheodory number of an antimatroid is 1 then its feasible sets are the ideals of a poset and hence the lattice is distributive (and therefore modular). Conversely, it is easy to see that for all other kinds of antimatroids, the lattice $\mathscr{L}(E, \mathscr{F})$ is non-modular.

We now prove that if $(E, \mathscr{F})$ has Caratheodory number at most two then $\mathscr{L}(E, \mathscr{F})$ is pseudomodular.

Let X and Y be any two feasible sets. Then by the definition of the rank and of $P_{X,Y}$, we have that

$$P_{X,Y} = \{Z \in \mathscr{F}:\ X \cap Y \subset Z \subset Y\}.$$

To show that X and Y have a pseudointersection, it suffices to verify the following (by Lemma 1.2(iii)): let Z, $Z-u$ and $Z-v$ be feasible sets with $X \cap Y \subset Z-u$, $Z-v$ and $Z \subset Y$, and let W be the largest feasible subset of $Z-u-v$; then $X \cap Y \subset W$. Suppose that this is not the case, then there exists an element $p \in (X \cap Y)-W$. Let $G = \{g \in E-W:\ W \cup \{g\} \in \mathscr{F}\}$. Then p is in the convex hull of G (this follows from properties (I) and (II) of antimatroids) and hence, by the definition of the Caratheodory number, we have a pair $\{q, r\} \subset G$ such that p is in the convex hull of $\{q, r\}$. Since p is an element in the feasible set $Z-u$, it follows that one of q and r must belong to $Z-u$. But none of q and r can belong to $Z-u-v$ since this would contradict the choice of W. Hence v must be one of q and r. Similarly, u must be one of q and r. But then X is a feasible set containing p but not q and r, which is a contradiction.

There are several important classes of antimatroids with Caratheodory number 2. We mention just a few:

Example 6a. Let E be any poset and let the convex sets be those sets which contain, along with any two comparable elements x, y, the whole interval $[x, y]$.

Example 6b. Let E be the vertex [edge] set of any tree T and let the convex sets be the vertex [edge] sets of subtrees.

Example 6c. Let E be any finite set in R^2 and let the convex sets be those subsets which contain, along with any two elements x and y, every point of E in the region of the plane bounded by the segment xy and by semilines pointing "upwards" from x and y.

3. Constructions preserving pseudomodularity

We show that some standard operations on semimodular lattices preserve pseudomodularity.

3.1. Direct product.

3.2. Truncation. For a semimodular lattice L and integer $k \geqq 1$, let $L_k = \{x \in L\colon r(x) < k \text{ or } x = 1\}$. Then the truncated lattice L_k is again semimodular, and it is easy to see that pseudomodularity is also preserved. The pseudointersection $x \urcorner_k y$ $(y \neq 1)$ in L_k is given by

$$x \urcorner_k y = \begin{cases} x \urcorner y, & \text{if } r(x \vee y) \leqq k \text{ in } L, \\ y, & \text{otherwise.} \end{cases}$$

A less trivial operation preserving pseudomodularity is the following:

3.3. Principal extension. Let L be a semimodular lattice and $w \in L - \{0\}$. The *principal extension* of L with respect to w is defined on the set

$$L' = L \cup \{y+p\colon y \in L,\ r(y \vee w) \geqq r(y)+2\}.$$

Here $y+p$ denotes a new element associated with the old lattice element y. The ordering is defined as before on the old elements, and by

$$\begin{aligned} x &\leqq y+p && \text{iff} \quad x \leqq y, \\ x+p &\leqq y+p && \text{iff} \quad x \leqq y, \\ x+p &\leqq y && \text{iff} \quad x \vee w \leqq y \end{aligned}$$

for $x, y \in L$. In particular it follows that $0+p$, which we denote shortly by p, is an atom and more generally, x is covered by $x+p$ whenever the latter exists. One can verify that L', with this partial ordering, is a semimodular lattice, containing L as a sublattice.

(This construction is best known for a geometric lattice, i.e., the lattice of flats of a matroid. Then the principal extension of L with respect to w means creating a new point p of the matroid which is "in general position" on the flat w.)

Theorem 3.4. *A principal extension of a pseudomodular lattice is again pseudomodular.*

Proof. The proof is more-or-less straightforward; nevertheless, we include it here for completeness. Let L be a pseudomodular lattice and $w \in L - \{0\}$. Let L' be the principal extension of L with respect to w. Observe that the class of "new" elements is closed under intersection: $(x+p) \wedge (y+p) = (x \wedge y)+p$, and so is of course the class of "old" elements. Further, if x is "old" and $y+p$ is "new" then $x\wedge(y+p)=x\wedge y$ if $p\not\leqq x$, and $x\wedge(y+p)=(x\wedge y)+p$ if $p\leqq x$.

We verify that condition (iv) of Theorem 1.4 holds for L'. Let x, y, z and u be elements of L' as in (iv). We may assume that they are distinct and that $x\wedge y\wedge z\neq x\wedge y$. The argument will be divided into several cases depending on the distribution of "new" elements among x, y, and z.

Case 1. x, y and z are "old". Then u also must be "old", and we know that (iv) is valid in L.

Case 2. $x=x_0+p$, $y=y_0+p$ and $z=z_0+p$ are "new" elements. Then condition (iii) applied to x_0, y_0 and z_0 within L, implies (iv) for x, y and z.

Case 3. $z=z_0+p$, and x, y are "old". Then we have the following subcases.

Subcase 3.1. $p\not\leqq x, y$. Then $x\wedge z$ is an "old" element covered by z and hence, $x\wedge z=z_0$. Similarly, $y\wedge z=z_0$ and the assertion is obvious.

Subcase 3.2. $p\leqq x$ but $p\not\leqq y$ (say). Then as before, $y\wedge z=z_0$ and hence $x\wedge y\wedge z=x\wedge z_0$. Since $x\wedge z=(x\wedge z_0)+p$, it follows that

$$r(x\wedge y) \leqq r(x)-1 = r(x\wedge z) = r(x\wedge z_0)+1 = r(x\wedge y\wedge z)+1,$$

which proves (iv).

Subcase 3.3. $p\leqq x, y$. Then $z\wedge x=(z_0\wedge x)+p$ is covered by $z=z_0+p$ by hypothesis, and hence $z_0\wedge x$ is covered by z_0. Similarly, $z_0\wedge y$ is covered by z_0. Since x is an "old" element above p, and u covers x, u must also be "old". Hence u, x, y and z_0 are elements of the old lattice L satisfying the conditions of Theorem 1.4 (v), and hence by the pseudomodularity of L, we obtain that

$$r(x\wedge y)-r(x\wedge y\wedge z_0) \leqq r(u)-r(z_0) = 2.$$

Since $x\wedge y\wedge z=(x\wedge y\wedge z_0)+p$, again (iv) follows.

Case 4. $x=x_0+p$, and z is "old". By symmetry this also handles the case when y is "new" and z is "old".

Subcase 4.1. $p\not\leqq z$. Then $x\wedge z$ is an "old" element covered by x and hence $x\wedge z=x_0$. So $x\wedge y\wedge z=x_0\wedge y$. Now $x\wedge y$ is either $x_0\wedge y$ or $(x_0\wedge y)+p$, which proves (iv).

Subcase 4.2. $p\leqq z$ and y is "old". Then $x\wedge z=(x_0\wedge z)+p$ is covered by $x=x_0+p$ by hypothesis, and hence $x_0\wedge z$ is covered by x_0. We can apply Theorem 1.4 (iii) to the "old" elements x_0, y and z and obtain that $r(x_0\wedge y)-r(x_0\wedge y\wedge z)\leqq 1$. Now, if $p\leqq y$ then $x\wedge y\wedge z=(x_0\wedge y\wedge z)+p$ and $x\wedge y=(x_0\wedge y)+p$; if $p\not\leqq y$ then $x\wedge y\wedge z=x_0\wedge y\wedge z$ and $x\wedge y=x_0\wedge y$. In either case, (iv) follows.

Subcase 4.3. $p \leqq z$ and $y = y_0 + p$. Then, as in the preceding case, $x_0 \wedge z$ is covered by x_0. Similarly, $y_0 \wedge z$ is covered by y_0. Apply (iii) to the elements x_0, y_0 and z, and obtain that $r(x_0 \wedge y_0) - r(x_0 \wedge y_0 \wedge z) \leqq 1$. Now, $x \wedge y \wedge z = (x_0 \wedge y_0 \wedge z) + p$ and $x \wedge y = (x_0 \wedge y_0) + p$, and we are done again.

Case 5. $x = x_0 + p$, $z = z_0 + p$, and y is "old". By symmetry this also handles the case when x is the only "old" element.

Subcase 5.1. $p \not\leqq y$. Then $y \wedge z$ is an "old" element covered by z, hence $y \wedge z = z_0$, and $x \wedge y \wedge z = x \wedge z_0 = x_0 \wedge z_0$. Since x covers $x \wedge z = (x_0 \wedge z_0) + p$, we get $r(x) - r(x_0 \wedge z_0) = 2$. So, $r(x \wedge y) \leqq r(x) - 1 = r(x \wedge y \wedge z) + 1$, and (iv) follows.

Subcase 5.2. $p \leqq y$. Since $x \wedge z = (x_0 \wedge z_0) + p$ and $y \wedge z = (y \wedge z_0) + p$, we have that $r(x) = r(x_0) + 1 = r(z_0) + 1 = r(x_0 \wedge z_0) + 2 = r(y \wedge z_0) + 2$. We may assume that $x_0 \wedge z_0 \not\leqq y$ (else $x \wedge z \leqq y$ and $x \wedge y \wedge z = x \wedge y$). Choose $t \in L$ so that $y \wedge z_0 < t < y$. Since $(x_0 \wedge z_0) \vee y = u$ covers y and $(x_0 \wedge z_0) \vee (y \wedge z_0) = z_0$ covers $y \wedge z_0$, it follows by semimodularity that $z' = (x_0 \wedge z_0) \vee t$ covers t. Clearly $z' \in L$, $r(z') = r(y) = r(t) + 1$, and $y \wedge z' = t$.

First, suppose that $x_0 \not\leqq z'$. Then $x_0 \wedge z' = x_0 \wedge z_0$, which is covered by x_0. Applying Theorem 1.4 (iii) to the "old" elements x_0, y, z', we obtain that $r(x_0 \wedge y) - 1 \leqq r(x_0 \wedge z' \wedge y) = r(x_0 \wedge z_0 \wedge y)$.

Second, suppose that $x_0 \leqq z'$. Since t covers $t \wedge z_0 = y \wedge z_0$, we may apply Theorem 1.4 (iii) to the elements x_0, t and z_0. This yields $r(x_0 \wedge t) - 1 \leqq r(x_0 \wedge z_0 \wedge t) = r(x_0 \wedge z_0 \wedge y)$. But $x_0 \wedge t = x_0 \wedge z' \wedge y = x_0 \wedge y$.

We have shown that in either case $r(x_0 \wedge y) - r(x_0 \wedge z_0 \wedge y) \leqq 1$. Since $x \wedge y = (x_0 \wedge y) + p$ and $x \wedge y \wedge z = (x_0 \wedge z_0 \wedge y) + p$, this proves (iv).

Observe that full transversal matroids, as defined in the previous section, can be obtained from Boolean algebras by principal extensions (infinitely often with respect to each flat). Hence the pseudomodularity of these matroids follows by an easy compactness argument. More generally, every matroid which can be obtained by principal extensions from Boolean algebras is pseudomodular. These matroids are all transversal, and can be represented as follows. Let G be a bipartite graph, and assume that one of its color classes S has r elements (the other may be finite of infinite). Also assume that for each $s \in S$, the other color class T contains an element which is connected only to s. Then the transversal matroid on T induced by G (in which a subset $T' \subseteqq T$ is independent iff G contains a matching covering T') is pseudomodular.

On the other hand, not every transversal matroid is pseudomodular: let $S = \{1, 2, 3, 4\}$, $T = \{a, b, c, d, e, f\}$, $V(G) = S \cup T$, and $E(G) = \{2a, 3b, 4c, 1d, 2d, 1e, 3e, 1f, 4f\}$. Then the transversal matroid induced by G on T is not pseudomodular (the flats *abde*, *acdf* and *bcef* violate condition (iv) of Theorem 1.4).

The fact that partition matroids are pseudomodular can be restated so that the Dilworth truncation of a Boolean algebra is pseudomodular. It is an interesting problem to find a broader class of lattices whose Dilworth truncations are pseudomodular.

4. Stretch embeddings and continuous matroids

We prove here the key theorem which will enable us to construct "stretch embeddings" and thereby continuous analogues of some classes of geometric lattices. This theorem generalizes a well-known result for modular lattices, see BIRKHOFF [1], pp. 73—74.

Theorem 4.1. *Let L be a pseudomodular lattice and $a_1, \dots, a_k$ elements of L such that $r(a_1)+\dots+r(a_k)=r(a_1\vee\dots\vee a_k)$. Then the sublattice generated by the intervals $[0, a_i]$ is isomorphic to the direct product of these intervals.*

Proof. Obviously, it suffices to consider the case $k=2$. Note that the submodularity of the lattice and the hypothesis that $r(a_1)+r(a_2)=r(a_1\vee a_2)$ imply that $r(x_1)+r(x_2)=r(x_1\vee x_2)$ for all $x_i\leqq a_i$.

Let L' be the sublattice generated by the intervals $[0, a_i]$. Define the mapping $\varphi(x_1, x_2)=x_1\vee x_2$. It is easy to see that this is an injection of $[0, a_1]\times[0, a_2]$ into L', and that this injection preserves joins. We will show that it also preserves meets. This will then also imply that the mapping is bijective.

Let $x_i, y_i\in[0, a_i]$ and set $p=(x_1\vee x_2)\wedge(y_1\vee y_2)$, $q=(x_1\wedge y_1)\vee(x_2\vee y_2)$. We want to show that $p=q$. It is obvious that $p\geqq q$. To show that equality holds, we show that p and q have the same rank. Clearly, $r(q)=r(x_1\wedge y_1)+r(x_2\wedge y_2)$.

To estimate $r(p)$, let $a=x_1$, $b=y_1$ and $c=x_2\vee y_2\vee p$ in Theorem 1.4 (ii). Then trivially $a\vee b\vee c=x_1\wedge y_1\vee x_2\vee y_2$ and hence

$$r(a\vee b\vee c) = r(x_1\vee y_1)+r(x_2\vee y_2).$$

Similarly we can compute that

$$r(a\vee b) = r(x_1\vee y_1),\quad r(a\vee c) = r(x_1)+r(x_2\vee y_2),\quad r(b\vee c) = r(y_1)+r(x_2\vee y_2).$$

This shows that a, b and c satisfy the conditions in Theorem 1.4 (ii), and hence by the pseudomodularity of L, we have

$$r(c) \leqq r(a\wedge b)+r(a\vee c)-r(a),$$

or, substituting,

$$r(x_2\vee y_2\vee p) \leqq r(x_1\wedge y_1)+r(x_2\vee y_2).$$

Interchanging the subscripts, we obtain

$$r(x_1\vee y_1\vee p) \leqq r(x_2\wedge y_2)+r(x_1\vee y_1).$$

Hence by submodularity,

$$r(p) \leqq r(p \vee x_1 \vee y_1) + r(p \vee x_2 \vee y_2) - r(x_1 \vee y_1 \vee x_2 \vee y_2) \leqq$$
$$\leqq r(x_1 \wedge y_1) + r(x_2 \wedge y_2) = r(q).$$

This proves the theorem.

It takes a little time to see that this theorem does not hold automatically in every semimodular or geometric lattice. Let Σ_1 and Σ_2 be two disjoint planes in a rank 6 projective space, and let e_i and f_i be two lines in Σ_i. Construct a matroid by deleting the intersection point of e_1 and f_1 as well as the intersection point of e_2 and f_2 from the space. Then in the lattice of flats of this matroid, $e_1 \wedge f_1 = e_2 \wedge f_2 = 0$, but $(e_1 \vee e_2) \wedge (f_1 \vee f_2) \neq 0$. This shows that at least the trivial mapping φ used in the proof above does not work. In fact, it is easy to see that this gives a counterexample.

The previous theorem enables us to construct "stretch embeddings" for various classes of matroids. Let $L_1, L_2, \ldots$ be a sequence of pseudomodular geometric lattices such that L_n has height n. Assume that for each $n, m \geqq 1$ such that $m|n$, there exist in L_n n/m elements $a_1, \ldots, a_{n/m}$ of rank m such that $a_1 \vee \ldots \vee a_{n/m} = 1$ and $[0, a_i] \cong L_m$. We call these elements the *representatives* of L_m in L_n.

It is now easy to define a *stretch embedding* of L_m in L_n, i.e., a lattice embedding $\varphi = \varphi_m^n \colon L_m \to L_n$ such that $r(\varphi(x)) = (n/m) r(x)$ for each $x \in L_m$. For, let $\varphi_i \colon L_m \to [0, a_i]$ $(i = 1, \ldots, n/m)$ be any isomorphism, and define $\varphi(x) = \varphi_1(x) \vee \ldots \vee \varphi_{n/m}(x)$. Theorem 4.1 implies that this is indeed a stretch embedding.

In the paper of BJÖRNER [2], a similar construction was described under the hypothesis that the elements $a_1, \ldots, a_{n/m}$ are modular. Since we assume the existence of pseudointersections for all pairs of elements, the construction in this paper is neither stronger nor weaker than that.

To construct the "continuous limit" of this sequence of geometric lattices, we have to assume that the mappings φ_m^n form a directed system, i.e., if $k|m$ and $m|n$ then $\varphi_m^n \circ \varphi_k^m = \varphi_k^n$. One may assure this by compatibly choosing the representatives. This was done for the partition lattices in BJÖRNER [2]; we describe below how such a choice can be made in the special families of matroids mentioned before.

Continuous algebraic matroids. Let F be an algebraically closed field. For each $n \geqq 1$, let K_n be an algebraically closed field extension of F of transcendence degree n, and let $L_n = \mathscr{L}(F, K_n)$. Let $\{x_1, \ldots, x_n\}$ be a transcendence basis. Let A_i be the algebraically closed subfield of K_n generated by $\{x_{(i-1)m+1}, \ldots, x_{im}\}$ $(i = 1, \ldots, n/m)$. Then $A_1, \ldots, A_{n/m}$ are appropriate representatives of L_m in L_n, and it is easy to check that the induced mappings form a directed system.

Continuous transversal matroids. Let $L_n = \mathscr{TM}(n)$ be the full transversal matroid of rank n, constructed in Section 2. Let $S=\{x_1, ..., x_n\}$. Assume that $m|n$ and let $S_i=\{x_{(i-1)m+1}, ..., x_{im}\}$ for $i=1, ..., n/m$. Then $\Omega(S_i)$ is a flat in L_n and these flats can be chosen as representatives of L_m in L_n. It is straighforward to check that the induced mappings form a directed system.

Now as in BJÖRNER [2], we can construct the direct limit $L_{(\infty)}$ of the system $\{L_k, \varphi_k^m\}$ and its completion L_∞, obtaining thereby continuous algebraic and transversal matroids. The study of these objects is, however, left to another paper.

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(A. B.)
DEPARTMENT OF MATHEMATICS
ROYAL INSTITUTE OF TECHNOLOGY
10044 STOCKHOLM, SWEDEN

(L. L.)
DEPARTMENT OF COMPUTER SCIENCE
EÖTVÖS LORÁND UNIVERSITY
1088 BUDAPEST, HUNGARY